Riemann integration as a limit of sum :—

We arrived at the integral of a function via the upper and the lower sums. The number M_i, m_i occurring in these sums may not be the value of the function f (In fact, M_i and m_i are values of f, if f is continuous). We shall now approach the integral $\int_a^b f(x)dx$ via the limit of a sequence of sums in which M_i and m_i are replaced by values of f.

Riemann Sum :—

Let f be a real valued function defined on $[a,b]$. Let $P: \{a = x_0, x_1, x_2 \dots x_n = b\}$ be a partition of $[a,b]$. Let ξ_i be any arbitrary point of $\delta_i = [x_{i-1}, x_i]$, $(i = 1, 2, 3, \dots n)$. Then the sum $\sum_{i=1}^{n} f(\xi_i)\delta_i$ is called a Riemann sum of f on $[a,b]$ relative to P. Since ξ_i is arbitary, therefore, corresponding to each partition P of $[a,b]$, there exist infinitely many Riemann sums.

We shall now give below another definition of a function to be integrable.

i) Let a bounded function f be integrable on $[a,b]$ according to the former definition, so that
$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx.$$

Let $\varepsilon > 0$ be given.

Then, by Darboux's theorem, there exists a $\delta > 0$ such that every partition P with $\|P\| < \delta$
$$U(f,P) < \overline{\int_a^b} f(x)dx + \varepsilon = \int_a^b f(x)dx + \varepsilon \quad \text{—— (1)}$$

and $L(f,P) > \int_a^b f(x)\,dx - \varepsilon = \int_a^b f(x)\,dx - \varepsilon$ —— (2)

Let $\xi_i \in \delta_i = [x_{i-1}, x_i]$, $(i = 1, 2, \dots, n)$ be arbitrary,

Then,

$$L(f,P) \leq \sum_{i=1}^{n} f(\xi_i)\delta_i \leq U(f,P) \quad \text{—— (3)}$$

$\therefore$ From (1) to (3), we have for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for each partition P, with $\|P\| < \delta$,

$$\int_a^b f(x)\,dx - \varepsilon < \sum_{i=1}^{n} f(\xi_i)\delta_i < \int_a^b f(x)\,dx + \varepsilon.$$

$$\Rightarrow \left|\sum_{i=1}^{n} f(\xi_i)\delta_i - I\right| < \varepsilon \text{, where } I = \int_a^b f(x)\,dx.$$

$\Rightarrow f$ is integrable according to the second definition.

ii) Conversely, let f be integrable on $[a,b]$ according to the second definition.

We shall show that f is bounded and its upper and lower integrals on $[a,b]$ are equal.

Let, if possible, f be not bounded on $[a,b]$.

By second definition, for $\varepsilon = 1$, there exists $\delta > 0$ and a number such that for each partition P of $[a,b]$ with $\|P\| < \delta$,

$$\left|\sum_{i=1}^{n} f(\xi_i)\delta_i - I\right| < 1, \ \forall\, \xi_i \in \delta_i.$$

$$\Rightarrow |I| - 1 < \sum_{i=1}^{n} f(\xi_i)\,\delta_i < |I| + 1, \ \forall\, \xi_i \in \delta_i \quad \text{—— (1)}$$

As f is not bounded on $[a,b]$, it must not be bounded on at least one sub-interval, δ_m (say).

Taking ξ_i = mid point of ith sub interval, when $i = m$, each from ξ_i except ξ_m is fixed and accordingly each term of $\sum_{i=1}^{n} f(\xi_i)\delta_i$, except $f(\xi_m)\delta_m$ is also fined. Since f is not bounded on S_m, we can choose a point (ξ_m) in S_m such that $\sum_{i=1}^{n} f(\xi_m)\,\delta_i > |I| + I$

which contradicts (1).

Hence, f is bounded on $[a, b]$. Let $\epsilon > 0$ be given, then there exists a $\delta > 0$ and a number I such that for every partition P with $\|P\| < \delta$,

$$\left| \sum_{i=1}^{n} f(\xi_i)\,\delta_i - I \right| < \epsilon/2, \forall \xi_i \in \delta_i .$$

$$\Rightarrow I - \epsilon/2 < \sum_{i=1}^{n} f(\xi_i)\,\delta_i < I + \epsilon/2, \forall \xi_i \in \delta_i - ②$$

If m_i and M_i are the g.l.b and l.u.b of f in $\delta_i = [x_{i-1}, x_i]$, then there exists points $\alpha_i, \beta_i \in \delta_i$ such that

$$f(\alpha_i) < m_i + \frac{\epsilon}{2(b-a)} \text{ and } f(\beta_i) < M_i - \frac{\epsilon}{2(b-a)}$$

$$\Rightarrow \sum_{i=1}^{n} f(\alpha_i)\,\delta_i < L(f, P) + \epsilon/2 \text{ and } \sum_{i=1}^{n} f(\beta_i)\,\delta_i > U(f, P) - \epsilon/2$$

— ③

∴ from ② and ③ taking $\xi_i = \alpha_i$ and β_i and have

$$I - \epsilon < L(f, P) \leq U(f, P) < I + \epsilon \quad — ④$$

For every partition P with $\|P\| < \delta$.

$$\text{Since } L(f, P) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(f, P) — ⑤$$

∴ From ④ and ⑤

$$I - \epsilon < \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq I + \epsilon .$$

$$\Rightarrow \left| \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \right| < 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, this implies.

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx .$$

$f(x)$ is integrable according to the former definition.

Least upper Bound :—

Let k be upper bound from set S for ever $\epsilon > 0$ there exists some $y \in S$ s.t $y > k - \epsilon$ then k is l.u.b of sets.

Greatest Lower Bound :—

If k is a lower bound for set S for every $\epsilon > 0$ there exists some $y \in S$ s.t. $y < k + \epsilon$. then k is glb of set S.

Theorem :—

If f is bounded function on (a, b) then for each are $\epsilon > 0$.

$f \in R[a, b]$ then there exists $\delta > 0$, s.t. $\int_a^b f(x)\,dx - L(f, P) < \epsilon$ or $L(f, P) > \int_a^b f(x)\,dx - \epsilon$ and

$U(f, P) - \int_a^b f(x)\,dx < \epsilon$ or $U(f, P) < \int_a^b f(x)\,dx + \epsilon$ for each partition P of $[a, b]$ with $\|P\| < \delta$.

Proof Given that $f \in R[a, b]$, $\Rightarrow$ for every $\epsilon > 0$ there exists a partition P on $[a, b]$ s.t $\|P\| < \delta$ —— (1)

$\Rightarrow U(f, P) - L(f, P) < \epsilon$.

Since $f \in R[a, b]$, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$.

Now $\inf\limits_{[a,b]} U(f, P) = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$.

$\therefore U(f, P) \geq \int_a^b f(x)\,dx$. —— (A)

$\because U(f, P)$ is a upper bound and $\sup\limits_{[a,b]} U(f, P) = \mathrm{lub}\,U(f, P)$ $[a, b]$

$\sup\limits_{[a,b]} L(f, P) = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \Rightarrow L(f, P) \leq \int_a^b f(x)\,dx$ —(2)

From (1) and (2)

$L(f,P) \leq \int_a^b f(x)\,dx \leq U(f,P)$ —— (B)

From (A) & (B) it follows that

$\int_a^b f(x)\,dx - L(f,P) < \epsilon$ and $U(f,P) - \int_a^b f(x)\,dx < \epsilon$.

From (1) and (2)

$\lim_{\|P\| \to 0} L(f,P) = \int_a^b f(x)\,dx$ and $\lim_{\|P\| \to 0} U(f,P) = \int_a^b f(x)\,dx$.

Integration by parts :-

In calculus, and more generally, in mathematical analyses, integration by parts or partial integration is a process that finds the integral of a product of function in terms of the integral of the product of their derivative and antiderivative. It is frequently used to transform the antiderivative of a product of functions into an antiderivative for which a solⁿ can be more easily found. The rule can be through of as an integral version of the product rule of differentiation.

If $u = u(x)$ and $du = u'(x)\,dx$ while $v = v(x)$ and $dv = v'(x)\,dx$ then the integration by parts formula states that

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,dx$$

$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx.$$

more compactly,

$$\int u\,dv = uv - \int v\,du$$

Mathematician Brook Taylor discovered integration by parts, first publishing the idea in 1715. More general formulations of integration by parts exist for the Riemann-Stieltjes and Lebesgue-Stieltjes integrals. The discrete analogue for sequences is called summation by parts.

Reduction Formula :—

1) To find some integrals we can use the reduction formulas. These formulas enable us to reduce the degree of the integrand and calculate the integrals in a finite number of steps. Below are the reduction formulas for integrals involving the most common function.

2) $\int x^n e^{mx} dx = \frac{1}{m} x^n e^{mx} - \frac{n}{m} \int x^{n-1} e^{mx} dx .$

3) $\int \frac{e^{mx}}{x^n} dx = - \frac{e^{mx}}{(n-1) x^{n-1}} + \frac{m}{n-1} \int \frac{e^{mx}}{x^{n-1}} dx , n \neq 1 .$

4) $\int \sinh^n x \, dx = \frac{1}{n} \sinh^{n-1} x \cosh x - \frac{n-1}{n} \int \sinh^{n-2} x \, dx$

5) $\int \frac{dx}{\sinh^n x} = - \frac{\cosh x}{(n-1) \sinh^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sinh^{n-2} x} , n \neq 1$

6) $\int \cosh^n x \, dx = \frac{1}{n} \sinh x \cosh^{n-1} x + \frac{n-1}{n} \int \cosh^{n-2} x \, dx$

7) $\int \frac{dx}{\cosh^n x} = - \frac{\sinh x}{(n-1) \cosh^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cosh^{n-2} x} , n \neq 1 .$

8) $\int \sinh^n x \cosh^m x \, dx = \frac{\sinh^{n+1} x \cosh^{m-1} x}{n+m} + \frac{m-1}{n+m} \int \sinh^n x \cosh^{m-2} x \, dx .$

Definite Integral :—

In this section we will formally define the definite and give many of the properties of definite integrals. Let's start off with the definition of a definite integral.

Given a function $f(x)$ that is continuous on the interval $[a,b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x .$$

Ex :- $\int_0^2 x^2+1\,dx$.

Properties :—

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

we can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.

2) $\int_a^b f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.

3) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.

4) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

we can break up definite integrals across a sum of difference.

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

where c is any number. This property is more important that we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals [a,c] and [c,d]. Note however that c doesn't need to be between a and b.

6) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

Integration by substitution :—

In calculus, integration by substitution, also known as u-substitution or change of variable is a method for evaluating integrals and antiderivatives. It is the counterpart to the chain rule for differentiation, and can loosely be thought of as using the chain rule "backwards"..

Substitution for a single variable :—

Before stating the result rigorously, consider a simple case using indefinite integrals.

Compute $\int (2x^3+1)^7 (x^2)\,dx$.

Set $u = 2x^3+1$. This means $\frac{du}{dx} = 6x^2$, or, in differential form $du = 6x^2 dx$. Now:

$$\int (2x^3+1)^7 (x^2)\,dx = \frac{1}{6}\int \underbrace{(2x^3+1)^7}_{u^7} \underbrace{(6x^2)\,dx}_{du} = \frac{1}{6}\int u^7\,du$$

$$= \frac{1}{6}\left(\frac{1}{8}u^8\right) + c = \frac{1}{48}(2x^3+1)^8 + c$$

where c is an arbitrary constant of integration.

This procedure is frequently used, but not all integrals are of a form that permits its use. In any event, the result should be verified by differentiating and comparing to the original integrand.

$$\frac{d}{dx}\left[\frac{1}{48}(2x^3+1)^8 + c\right] = \frac{1}{6}(2x^3+1)^7(6x^2) = (2x^3+1)^7(x^2).$$

For definite integrals, the limits of integration must also be adjusted, but the procedure is mostly the same.

Substitution for multiple variables:—

One may also use substitution when integrating functions of several variables. Here the substitution function $(v_1, \ldots, v_n) = \phi(u_1, \ldots, u_n)$ needs to be injective and continuously differentiable, and the differentials transform as

$$dv_1 \cdots dv_n = |\det(D\varphi)(u_1, \ldots, u_n)|\, du_1 \cdots du_n$$

where $\det(D\varphi)(u_1, \ldots, u_n)$ denotes the determinant of the jacobian matrix of partial derivatives of φ at the point $(u_1, \ldots u_n)$. This formula expresses the fact that the absolute value of the determinant of a matrix equals the volume of the parallelotope spanned by its columns or rows.

more precisely, the change of variables formula is stated in the next theorem.

Theorem :—

Let U be an open set in R^n and $\varphi : U \to R^n$ an injective differentiable function with continuous partial derivatives, the jacobian of which is nonzero for every x in U. Then for any real-valued, compactly supported, continuous function f, with support contained in $\varphi(U)$,

$$\int_{\varphi(U)} f(v)\,dv = \int_U f(\varphi(U))\,|\det(D\varphi)|$$

The conditions on the theorem can be weakened in various ways. First, the requirement that φ be continuously differentiable can be replaced by the weaker assumption that φ be merely differentiable and have a continuous invers. This is guarantted to hold if φ is continuously differentiable by the inverse function theorem. Alternatively, the requirement that $\det(D\varphi) \neq 0$ can be eliminated by applying Sard's theorem.

For Lebesgue measurable functions, the theorem can be stated in the following form.